Splitting methods for nonlinear diffusion filtering

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Abstract

A PDE based method for image restoration is described by the heat equation with Neumann and initial conditions, where diffusivity is chosen as a rapidly decreasing function of the gradient magnitude of the solution and the initial condition represents the observed image. A widely used method for the numerical solution of this parabolic equation is the \( \theta \)-method, that leads, at each time level, to a solution of a system of nonlinear equations of the form \((I - \tau A(u))u = w\), where the matrix \(A(u)\) satisfies irreducibility, vanishing column sum, negativity of diagonal entries and nonnegativity of the off-diagonal entries, Lipschitz continuity and symmetry. In this work we consider the application of splitting methods to the semi-implicit \( \theta \)-method and to the implicit \( \theta \)-method. In the case of semi-implicit \( \theta \)-method, the Arithmetic Mean method is considered for the solution of the linear system that occurs at each time level; this method has within its overall mathematical structure certain well defined substructures that can be executed simultaneously in order to increase the degree of multiprogramming. For the solution of the nonlinear system that occurs at each time level of the implicit \( \theta \)-method, multiplicative and additive operator splitting methods are examined. Because of the noncommutativity of the matrices of the splitting, the order of applying these matrices in the multiplicative operator splitting methods, as the Alternating Direction Implicit or the Fractional Step methods, can affect the final result in image denoising. Indeed, the filtered two-dimensional image will not be the same after a rotation of 90 degrees. A symmetric strategy does not suffer from this deficiency; this motivates the advantage of using additive operator splitting methods in image restoration problems.

Keywords: splitting methods, finite differences, image restoration.

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1 Statement of the problem

In recent years, partial differential equations (PDEs) have become popular and useful tools for digital image restoration, digital image segmentation and computer vision. A PDE–based method for image restoration, initially proposed by Perona and Malik ([19]), is described by the equation

\[ \frac{\partial u}{\partial t} = \text{div} (g(u) \nabla u) \]  

(1)

where the he function \( u \equiv u(x, y, t) \) satisfies on the boundary \( \partial R \) of the rectangular domain \( R \) the Neumann condition that the normal derivative is zero

\[ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial R \]  

\( \nu \) is the normal to \( \partial R \)  

(2)

and in \( \bar{R} \) the initial condition

\[ u(x, y, 0) = u_0(x, y) \]  

(3)

and the diffusivity \( g \) is chosen as a rapidly decreasing function of the gradient magnitude \( |\nabla u|^2 = (\frac{\partial u}{\partial x})^2 + (\frac{\partial u}{\partial y})^2 \) (edge indicator) and \( u_0(x, y) \) is the observed image (the data).

The time \( t \) is a scale parameter: increasing \( t \) leads to simpler image representations. The whole embedding of the original image into such a one–parameter family of simplified images is called \textit{scale–space}. For \( g = 1 \) we obtain the parabolic linear heat equation with the explicit solution ([3])

\[ u(x, y, t) = \int \int_{\mathbb{R}^2} K_{\sqrt{2}t}(x - \xi, y - \eta)u_0(\xi, \eta)d\xi d\eta = (K_{\sqrt{2}t} \otimes u_0)(x, y) \]  

(4)

where \( K_\sigma \) denotes the two–dimensional Gaussian kernel

\[ K_\sigma(x, y) = \frac{1}{(\sqrt{2\pi}\sigma)^2} \exp(-\frac{x^2 + y^2}{2\sigma^2}) \]  

(5)

Convolution by a positive kernel is the basic operation in linear image filtering. It corresponds to \textit{low–pass filtering} that inhibits high frequencies (oscillations in the space domain).

The solution of the heat equation satisfies an \textit{oversmoothing property}, for which the edges in the images can be lost or severely blurred. Thus, it is necessary to introduce a nonlinear diffusion model for removing the noise while preserving the edges at best.

In the nonlinear forward diffusion process (1)–(3) the space of \textit{functions of bounded variation} \(^1\) is the proper class for many basic image processing tasks.

\(^1\)See e.g. [2, p. 128].
In this case a typical choice of the diffusity $g$ is [24]

$$g(|\nabla u|^2) = \left( \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 + \beta^2 \right)^{-\frac{1}{2}}$$

(6)

where $\beta$ is a small enough positive artificial parameter used in the numerical regularization.

With the assumption (6) the nonlinear diffusion model (1)–(3) acts as a forward parabolic equation smoothing homogeneous regions while preserving the edges. A nonlinear diffusion problem can also be obtained by considering the diffusivity function $g(u)$ as in [26]

$$g(|\nabla u_\sigma|^2) = 1 - \exp\left(\frac{-3.315}{(|\nabla u_\sigma|^2/\lambda)^4}\right)$$

where $u_\sigma$ is obtained by convolving $u$ with the Gaussian kernel $K_\sigma$ and $\sigma$ is a regularization parameter for the well–posedness of the process ([6]). The parameter $\lambda$ describes the contrast: we have the diffusivity close to 0 for $|\nabla u_\sigma|^2 > \lambda$ (the edges) and the diffusivity close to 1 for $|\nabla u_\sigma|^2 < \lambda$ (interior points).

In the monography [3] different types of functions $g$ have been analysed (see also [16]). The mathematical framework for the existence and uniqueness of a solution of problem (1)–(3) with these different diffusivity functions has been clearly presented. Also the connections between scale–space analysis and partial differential equations has been concisely presented.

If the domain $R$ is the unit square, the spatial grid $R_h$ has grid–points $(x_i, y_j)$, $(i, j = 1, \ldots, N)$ with $x_i = x_{i-1} + h$ and $y_i = y_{i-1} + h$ for $i = 2, \ldots, N$, with $x_1 = y_1 = h/2$ and $x_{N+1} = x_N = x_N + h/2$, $y_{N+1} = y_N + h/2$.

The discretization by box–integration method ([22, §6.3, p. 204]) for the spatial term in equation (1), leads to the solution of the system of $n = N \times N$ ordinary differential equations of the type

$$\begin{cases}
\frac{d u(t)}{dt} & = A(u)u(t) & 0 < t \leq T \\
u(0) & = u_0
\end{cases}$$

(7)

where $u_0$ is a vector of $\mathbb{R}^n$ and it is the restriction of $u(x, y, 0)$ on $R_h$; the matrix $A(u) \equiv (a_{ij}(u))$ is a real $n \times n$ matrix and the vector $u(t)$ is the restriction of the solution $u(x, y, t)$ on $R_h$.

For natural ordering of the grid–points (by horizontal lines), $A(u)$ is the block tridiagonal matrix of order $n = N \times N$, where each diagonal block is a tridiagonal matrix of order $N$ and each subdiagonal and superdiagonal block is a diagonal matrix of order $N$.

Using only one index for numbering the grid points $P_i$, $i = 1, \ldots, n$, the elements
The elements $a_{ij}(u)$ of the matrix $A(u)$ are given by

$$a_{i,j}(u) = \begin{cases} \frac{g_i + g_j}{2h} & j \in \mathcal{N}(i) \\ -\sum_{k \in \mathcal{N}(i)} \frac{g_i + g_k}{2h} & j = i \\ 0 & \text{otherwise} \end{cases}$$

where $g_i$ denotes an approximation to $g(|\nabla u|^2)$ with $|\nabla u|^2$ evaluated at the grid points $P_i$ corresponding to the five–points difference stencil. Boundary points have only inner points as neighbours, as a result of the boundary condition (2).

For example an edge–preserving approximation to the gradient magnitude at an inner grid–point $P_i$ commonly used is given by the following central (second order) finite difference scheme

$$|\nabla u|^2_{P_i} \simeq \left( \frac{u_{i+1}(t) - u_{i-1}(t)}{2h} \right)^2 + \left( \frac{u_{i+N}(t) - u_{i-N}(t)}{2h} \right)^2$$

The expression remains also valid at boundary grid–points, if we extend the image $u$ by reflecting it at the boundary. Thus, in (5), $g_i = g(|\nabla u|^2_{P_i})$.

The matrix $A(u)$ satisfies the following properties:

C1) irreducibility for all $u \in \mathbb{R}^n$ (see e.g. [22, p. 18]);

C2) negative diagonal entries and nonnegative off–diagonal entries for all $u \in \mathbb{R}^n$;

C3) symmetry:

$$a_{i,j}(u) = a_{j,i}(u) \quad \text{for } i, j = 1, ..., n \text{ and all } u \in \mathbb{R}^n;$$

C4) vanishing row (column) sums:

$$\sum_{j=1}^{n} a_{i,j}(u) = 0 \quad \text{for } i = 1, ..., n \text{ and all } u \in \mathbb{R}^n;$$

C5) Lipschitz–continuity of $A(y)$ for every bounded subset of $\mathbb{R}^n$:

$$\|A(u) - A(v)\| \leq \Lambda \|u - v\|$$

for all $u, v$ of subset.

It is well known that the problem is well posed and the solution $u(t)$, for $t \to \infty$, tends to a constant steady–state (cfr., e.g. [25, §3.2, p. 76]).
2 Operator splitting methods

The matrix $A(u)$ in (7) can be decomposed in the form

$$A(u) = A_1(u) + A_2(u)$$

(10)

where $A_1(u)$ and $PA_2(u)P^T$ are two tridiagonal matrices and $P$ is the permutation matrix which reorders per columns the grid–points, i.e. the vector $u$ as $Pu$, that is:

$$u = (u_{11}, ..., u_{1N}, ..., u_{N1}, ..., u_{NN})^T; Pu = (u_{11}, ..., u_{1N}, ..., u_{N1}, ..., u_{NN})^T$$

Also the matrices $A_1(u)$ and $A_2(u)$ satisfy the properties C1)–C5). The matrices $A(u)$, $A_1(u)$ and $A_2(u)$ are symmetric semidefinite matrices for all $u \in \mathbb{R}^n$.

C1)–C5). The numerical solution of the initial value problem (7) is obtained by a step-by-step method. Widely used are the $\theta$–method (see e.g. [21]) and the operator splitting methods (see e.g. [14]).

The $\theta$–method is described by the following system ($k = 0, 1, ...$):

$$\left(I - \theta \Delta t A(u^{(k+1)})\right) u^{(k+1)} = \left(I + (1 - \theta) \Delta t A(u^{(k)})\right) u^{(k)}$$

(11)

$$u^{(0)} = u_0$$

where $\theta$ is a real parameter such that $0 \leq \theta \leq 1$; for any $\theta \neq 0$, the method (11) is implicit.

Here, $u^{(k)} \in \mathbb{R}^n$ is an approximation to $u(t)$ at the discrete value of time $t = k\Delta t$, where $\Delta t$ is the time step and $k$ is the time level, $k = 0, 1, ...$.

In the linear case, that is, when the elements of $A$ are not depending on $u^{(k)}$, the method (11) is first and second order accurate in time for $\theta = 1$ and $\theta = \frac{1}{2}$, respectively.

Thus, at each time level $k$ we have to solve a system of nonlinear equations of the form

$$\left(I - \tau A(u)\right) u = w$$

(12)

where $w \in \mathbb{R}^n$ is a specified source term and $\tau = \theta \Delta t > 0$. Here, we have dropped the indices $k$ and $k+1$ on $u$ for a simpler presentation.

The matrix $(I - \tau A(u))$ is an irreducibly diagonally dominant M–matrix [22, p. 23, p. 91].

A solution $u^*$ of the system (12) is a root of the nonlinear equations

$$F(u) = (I - \tau A(u)) u - w = 0$$

(13)

The questions of existence and uniqueness of the solution for (12) has been studied by many investigators. For example, in [8] an existence and uniqueness theorem about the solution of (13) has been established under the assumptions that the components of the vectors $u$ and $w$ are nonnegative, i.e. $u \in \mathbb{R}_+^n$, $w \in \mathbb{R}_+^n$. Here, we recall this result.
Theorem 1. Let \( F : \mathbb{R}^n_+ \rightarrow \mathbb{R}^n_+ \) be a mapping of the form (13) on which the conditions C1)–C5) hold for the matrix \( A(u) \) for all \( u \in \mathbb{R}^n_+ \) and the source term \( w \) is nonnegative: \( w \geq 0 \).

Let \( \Omega \subset \mathbb{R}^n_+ \) be the subset

\[
\Omega = \{ u \in \mathbb{R}^n_+ : \|u\|_1 = \|w\|_1 \}
\]

Then, the system (13) has a solution \( u^* \in \Omega \).

Furthermore, if \( \tau \Lambda \|w\|_1 < 1 \), this solution is unique.

(The norm \( \| \cdot \|_1 \) will be either the \( l_1 \)-norm for vectors or the norm that it induces for matrices).

This theorem suggests the use of the mapping

\[
u = G(u) \quad \text{with} \quad G(u) = (I - \tau A(u))^{-1}w
\]

for all \( u \in \mathbb{R}^n_+ \), as an iteration mapping to approximate the solution of the nonlinear system (13) [12, p. 66], [17, p. 153].

The procedure looks like:

set \( z^{(0)} = w; \rho > 0 \)

\[
\text{for } \nu = 0, 1, \ldots \text{ until convergence do }
\]

\[
z^{(\nu+1)} = (I - \tau A(z^{(\nu)}))^{-1}w
\]

If we denote with \( \nu^* \) the index of the iteration such that the convergence condition is satisfied, then \( u^{(k+1)} = z^{(\nu^*)} \).

Using (13), formula (16) can be written in the form, \( k = 0, 1, \ldots \):

\[
z^{(\nu+1)} = (I - \tau A(z^{(\nu)}))^{-1}w
\]

\[
= (I - \tau A(z^{(\nu)}))^{-1}((I - \tau A(z^{(\nu)}))z^{(\nu)} - F(z^{(\nu)}))
\]

\[
= z^{(\nu)} - (I - \tau A(z^{(\nu)}))^{-1}F(z^{(\nu)})
\]

Thus, the iterative method (16) can be viewed as a Newton–like method, where the matrix \( (I - \tau A(z^{(\nu)})) \) is an approximation to the jacobian matrix \( F'(z^{(\nu)}) \) (see e.g. [11]).

The assumption \( \tilde{\Lambda} = \tau \Lambda \|w\|_1 < 1 \) assures the convergence of the method (16) and an estimate of \( \tilde{\Lambda} \) is given by

\[
\tilde{\Lambda} \simeq \tilde{\Lambda}_\nu \equiv \|\Delta z^{(\nu)}\|_1 / \|\Delta z^{(\nu-1)}\|_1
\]

where \( \Delta z^{(\nu)} = z^{(\nu+1)} - z^{(\nu)} \).

This estimates is used for defining the convergence test of the iterative process (16) [17, p. 153].

Convergence of the sequence \( \{z^{(\nu)}\} \) to a zero \( u^* \) of the function \( F(u) \) can also be shown and checked in practice under the following \textit{a–posteriori} conditions.
A–posteriori convergence test. Let $\Omega \subset \mathbb{R}^n$ be a compact set and $F : \Omega \rightarrow \mathbb{R}^n$ be a continuous mapping. Consider a sequence $\{z^{(\nu)}\}$ in $\Omega$ satisfying
\[
\|\tilde{z}^{(\nu+1)} - \tilde{z}^{(\nu)}\|_1 \leq \vartheta_{\nu} \|\tilde{z}^{(\nu)} - \tilde{z}^{(\nu-1)}\|_1 \vartheta_{\nu} \leq \bar{\vartheta} < 1
\]
\[
\|F(\tilde{z}^{(\nu+1)})\|_1 \leq \vartheta_{\nu} \|F(\tilde{z}^{(\nu)})\|_1 \vartheta_{\nu} \leq \bar{\vartheta} < 1
\]
Then, the sequence $\{z^{(\nu)}\}$ converges in $\Omega$ to a zero $u^*$ of $F$. For the computation it is no restriction to assume the entire iterative process remains in a compact, convex set $\Omega \subset \mathbb{R}^n$ ([20, p. 71]).

At each stage of the iterative method (16), the linear system (17), that is
\[
(I - \tau A(z^{(\nu)}))\Delta z = -F(z^{(\nu)})
\]
has to be solved. The computation of the exact solution $\Delta z^{(\nu)}$ can be too expensive if $n$ is large and, for any $n$, may not be justified when $z^{(\nu)}$ is relatively far from $u^*$. Therefore, one might prefer to compute some approximate solution of the system (19).

There exists a considerable body of literature on the solvability of (symmetric and nonsymmetric) linear system of the form (19), in which the coefficient matrix is an M–matrix. The iterative method (16), combined with the conjugate gradient method for solving (19) is quite popular for nonlinear PDEs (1)–(3) in image processing.

Because of the phenomenon of semiconvergence of the conjugate gradient method, the relative restoration error decreases monotonically at the first few iterations and increases in later stages of the iterations. If the starting vector is zero, the solution norm of the conjugate gradient method increases monotonically and the residual norm decreases monotonically with the iteration number. The monotonic behaviour of both these vectors is important in connections with the stopping criterion for the regularizing the conjugate gradient iterations (see [7, Chapt. 6]).

Furthermore, the adaptability of method (16) with the conjugate gradient solver for system (19) to parallel computer has been extensively analysed.

Finally, we remark that when the diffusivity $g$ has the form (6), the determination of the solution of (13) with the Newton’s method is not advantageous, because the domain of convergence of the method is extremely small, especially when the regularization parameter $\beta^2$ is small.

If we perform only one iteration $\nu$ in the process (16) with $\theta = 1$ and $z^{(0)} = u^{(k)}$, we have the semi–implicit method
\[
(I - \Delta t A(u^{(k)}))u^{(k+1)} = u^{(k)}
\]
where the matrix $(I - \tau A(u^{(k)}))$ is a consistent approximation of $\exp(\Delta t A(u^{(k)}))$ ([22, p. 293]).

Recently additive operator splitting methods have been applied for solving nonlinear diffusion equations in image degradation. In the paper [26] the problem (1)–(3) with diffusivity $g$ of type (6) has been solved by the following Additive
Operator Splitting (AOS) method

\[ u^{(k+1)} = \frac{1}{2} \left[ \left( I - 2\Delta t A_1(u^{(k)}) \right)^{-1} + \left( I - 2\Delta t A_2(u^{(k)}) \right)^{-1} \right] u^{(k)} \]  

(21)

where \( A_1(u) \) and \( A_2(u) \) are the splitting (10) of \( A(u) \).

The matrix

\[ \frac{1}{2} \left[ \left( I - 2\Delta t A_1(u^{(k)}) \right)^{-1} + \left( I - 2\Delta t A_2(u^{(k)}) \right)^{-1} \right] \]

(22)

is a consistent approximation of \( \exp(\Delta t A(u^{(k)})) \) and (21) is a first order method in time.

For dissipative systems (7) (i.e. the symmetric matrices \( A + A^T \), \( A_1 + A_1^T \) and \( A_1 + A_1^T \) are negative semidefinite) the matrix (22) is stable [22, p. 290].

Unlike the Alternating Direction Implicit (ADI) method [18]

\[
\begin{cases}
(I - \frac{\Delta t}{2} A_1(u^{(k)})) u^{(k+1/2)} = (I + \frac{\Delta t}{2} A_2(u^{(k)})) u^{(k)} \\
(I - \frac{\Delta t}{2} A_2(u^{(k)})) u^{(k+1)} = (I + \frac{\Delta t}{2} A_1(u^{(k)})) u^{(k+1/2)}
\end{cases}
\]

(23)

or the Fractional Step (FS) method [27]

\[
\begin{cases}
(I - \frac{\Delta t}{2} A_1(u^{(k)})) \tilde{u} = A(u^{(k)}) u^{(k)} \\
(I - \frac{\Delta t}{2} A_2(u^{(k)})) \hat{u} = \hat{u} \\
u^{(k+1)} = u^{(k)} + \Delta t \hat{u}
\end{cases}
\]

(24)

or Local One Dimensional (LOD) methods ([15]), such as

\[
\begin{cases}
(I - t_1 A_1(u^{(k)})) \tilde{u} = u^{(k)} \\
(I - t_2 A_2(u^{(k)}))u^{(k+1)} = \hat{u}
\end{cases}
\]

(25)

which are multiplicative operator splitting methods, the AOS method (21) is symmetric and unconditionally satisfies all discrete diffusion scale-space requirements proposed in [26].

Because of the noncommutativity of the matrices \( A_1 \) and \( A_2 \), the order of applying these matrices in the multiplicative operator splitting methods can affect the final result in image denoising. Indeed, the filtered two-dimensional image will not be the same after a rotation of 90 degrees. A symmetric strategy does not suffer from this deficiency. This motivates the use of additive operator splitting methods in image restoration problems.

In the papers [10] and [9], the method of Arithmetic Mean has been proposed; this is an additive–multiplicative operator splitting method and it is ideally suited for an implementation on parallel computers. We can write this method for system (7) in the form
\[
\begin{align*}
\tilde{u} &= R_1^{(k)} R_2^{(k)} u^{(k)} \\
\hat{u} &= R_2^{(k)} R_1^{(k)} u^{(k)} \\
u^{(k+1)} &= \frac{1}{2} (\tilde{u} + \hat{u})
\end{align*}
\] (26)

where \( R_i^{(k)} (i = 1, 2) \) has the expression
\[
R_i^{(k)} = (I - \Delta t A_i(u^{(k)}))^{-1}
\] (27)
or
\[
R_i^{(k)} = \left( I - \frac{\Delta t}{2} A_i(u^{(k)}) \right)^{-1} \left( I + \frac{\Delta t}{2} A_i(u^{(k)}) \right)
\] (28)
or
\[
R_i^{(k)} = \left( I - \alpha_1 \Delta t A_i(u^{(k)}) \right)^{-2} \left( I + \alpha_2 \Delta t A_i(u^{(k)}) \right)
\] (29)

with \( \alpha_2 = \sqrt{2} - 1 \) and \( \alpha_1 = \alpha_2 / \sqrt{2} \).

With the forms (27), (28) and (29) of \( R_1^{(k)} \) and \( R_2^{(k)} \), the method (26) is consistent with the system (7); besides, it is first order accurate in time when we consider the formula (27) and second order accurate in time when we consider formulae (28) or (29). For dissipative systems (7) the methods (26)–(27), (26)–(28) and (26)–(29) are absolutely stable for all \( \Delta t \).

In the paper [4], the methods (26)–(27) and (26)–(28) have been applied for solving problem (1)–(3) with diffusivity \( g \) of type (6). A comparison between these methods and AOS method has been carried out on synthetic images in which the noise is an error to be eliminated and all piecewise constant textures must be perfectly preserved. The method (26)–(28) turns out to restore images with the best clarity and efficiency.

Finally we briefly point out the connection between the solution of system (20) and Tikhonov regularization.

It is well known that an image restoration process can be described by a Fredholm first–kind integral equation ([1], [13])
\[
\mathcal{H} v = g
\]

where \( \mathcal{H} \) is a linear operator in Hilbert spaces and the discrete approximate solution can be obtained by the Tikhonov regularization method, by solving the system
\[
(H^T H + \alpha I)v_\alpha = H^T g
\] (30)

where \( H \) is the discretization of the integral operator, \( g \) is the data (the observed image) and \( \alpha \) is a regularization parameter. See [7, Chapt. 7] and [23, Chapt. 7] for the methods for computing \( \alpha \).

Now, we consider the linear dynamic system:
\[
\frac{dv(t)}{dt} + H^T H v(t) - H^T g = 0 \quad t > 0
\] (31)
The eigenvalues of the matrix $-H^TH$ are negative, then the matrix $-H^TH$ is stable and
\[
\lim_{t \to \infty} v(t) = (H^TH)^{-1}(H^Tg) \equiv v^*.
\]
If we apply implicit Euler method to the system (31), we can write
\[
(H^TH + \frac{1}{\Delta t}I)v^{(k+1)} = \frac{1}{\Delta t}v^{(k)} + H^Tg
\]
(32)
where $\Delta t$ is the time step and $v^{(k)}$ is the approximation of the solution at $t = k\Delta t$. If we choose $v^{(0)} = 0$, at the first time step we have
\[
(H^TH + \frac{1}{\Delta t}I)v^{(1)} = H^Tg
\]
that is $v^{(1)}$ is the solution of (30) $v_\alpha$ with $\alpha = 1/\Delta t$. Instead of determining a solution of (30) with $\alpha$ small, that is $\Delta t$ large, we compute the solution $v^*$ with prespecified time step $\Delta t$ and we proceed with the iterations (32) (Successive Approximation Method with Smoothing [5]).

References


